# APPLICATION OF SIMILARITY THEORY TO FORECASTING THE MIXED-LAYER DEPTH OF THE OCEAN

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Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE

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## ABSTRACT

The thermal structure of the ocean, especially the uppermost mixed layer, greatly affects sonar ranges. In this paper, similarity theory is applied to the problem of forecasting the depth of the mixed layer during the warm season, assuming the controlling processes are secular, non-advective, and non-divergent. The resulting forecast method consists mainly of two equations. Parameters used are wind, coriolis effect, the coefficient of thermal expansion and a measure of the excess heat within the mixed layer. The constants in the equations were determined using data from OWS Papa (50N, 145W). The forecast method treats both seasonal and transitional thermoclines. The method was tested with data from OWS Papa and OWS November (30N, 140W). The tests apparently indicate wide applicability of this forecast method and thus tend to corroborate the proposal by Kitaigorodsky that the mixed-layer depth is a function of a universal coefficient.

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## LIST OF SYMBOLS AND ABBREVIATIONS

| | |
|---|---|
| $\beta$ | coefficient of thermal expansion |
| MLD | mixed layer depth |
| Q | Excess heat present in upper layer of water |
| $Q_F$ | Net heat flow to and from upper layer of water |
| $Q_T$ | Excess heat present in upper layer of water associated with a transitional thermocline |
| $Q_S$ | Excess heat present in upper layer of water associated with a seasonal thermocline |
| TS | Temperature at the surface of the ocean |
| W | wind speed or a representative wind speed |
| f | coriolis parameter |
| $\Omega$ | modified coriolis parameter $(f \times 10^4)$ |
| $\omega$ | angular velocity of earth |
| $\phi$ | latitude |

## 1. Introduction.

The detection of enemy submarines by sonar is one of the major unsolved problems of this decade. Despite recent technological advances in many fields of science, the nuclear submarine is still practically invulnerable. Even though sonar technology, in particular, has advanced rapidly, we are still unable to position our ASW forces or their sonars for optimum performance. This is largely because sonar ranges are either enhanced or reduced by refraction of sound energy in the surface layers and we are not able to forecast accurately the thermal structure which greatly determines refraction.

Many authors have devised systems for forecasting the thermal structure of the upper layer of the ocean based on either dynamical analysis or empirical relationships. The equations resulting from dynamical analysis are either too complicated, if all the physical processes are considered; or are impractical, if many simplifying assumptions are made. Empirical relations have been determined for certain locations and for limited time periods but do not appear to be valid universally.

Another possible method for developing a forecasting system uses similarity theory. Kitaigorodsky [1] investigated the application of similarity theory to the ocean thermal-structure-forecasting problem. Although his results have certain drawbacks, as noted in appendix IV, still he has shown the applicability of the method. Consequently, the method used by Kitaigorodsky is also applied by this author, with some modifications of parameters, in an effort to develop a more practical result. The form of P, a dimensionless coefficient inherent in the application of similarity theory, is determined by use of data from

Ocean Weather Ship Papa (50N 145W).

Figure 1 depicts a typical ocean thermal structure in the warm season at OWS Papa. Characteristically, there is a quasi-isothermal layer that extends from the surface to the upper boundary of a negative temperature gradient.

The depth to which the quasi-isothermal layer extends is usually referred to as the mixed-layer depth. In this paper the mixed-layer depth (MLD) is defined as that depth at which the temperature of the water first becomes 1C less than that of the water at the surface.

A transitional layer of negative temperature gradient between layers of relatively small temperature variation is referred to as a thermocline. Four main types of thermocline can be classified, primarily according to degree of permanence: diurnal, transitional, seasonal, and permanent.

Diurnal thermocline: it results from a net heat gain during the day, a small thermocline ($\Delta T < 1C$, see fig. 1) being formed close to the surface by late afternoon. With a net heat loss at night, the thermocline will be destroyed by morning.

Transitional thermocline: a moderately large thermocline ($\Delta T > 1C$) is formed when diurnal heat input exceeds losses. After a few days or weeks it joins with the seasonal thermocline as the added heat diffuses downward.

Seasonal thermocline: the transition zone which lies between the surface waters warmed during the summer and the colder water below. At OWS Papa the seasonal thermocline depth is about 20 meters in the late summer, and about 50 meters in the spring and fall. It is not present in the winter.

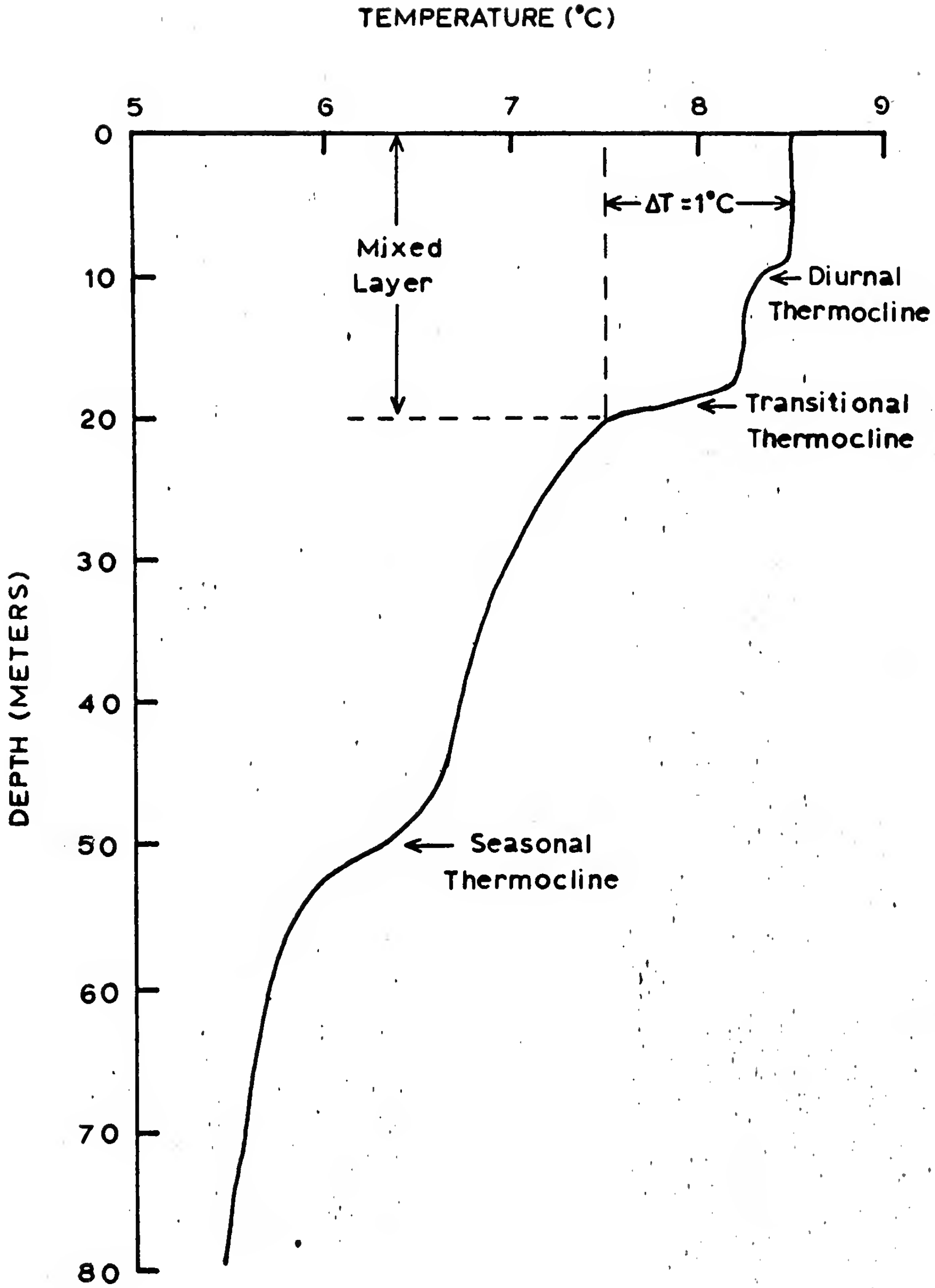


Figure 1

Typical Thermal Structure of the Upper Layer of the Ocean During the Warm Season at OWS Papa

Permanent thermocline: in many localities a deep thermocline that is relatively persistent in depth exists throughout the year. Below a permanent thermocline the temperature gradient (usually negative) is very small all the way to the bottom.

The entire thermal structure of the ocean, though of interest, is not immediately as important to sound propagation with present sonars as the depth of the uppermost mixed-layer of the ocean. As far as this mixed layer is concerned, there are two basic seasons at OWS Papa. These are the warm season, when mixing of the upper layer is mainly due to the wind, and the cool season, when mixing is mainly due to thermohaline convection. Thus the physical processes during the two seasons are quite different and are usually treated separately.

As in the paper by Kitaigorodsky, similarity theory will be applied to develop a method for forecasting the mixed-layer depths associated with the transitional and seasonal thermoclines during the warm season at OWS Papa. Henceforth, diurnal thermoclines will be ignored and the terms "MLD" and "depth of the thermocline" will be synonymous and will refer only to mixed-layer depths associated with either transitional or seasonal thermoclines. Only those changes in the MLD which are secular, non-advective, and non-divergent will be considered.

2. Description of Similarity Theory.

There are three basic steps in the application of similarity theory.

1) Determine the physical processes that control the physical phenomenon of interest.

2) Select parameters that accurately represent the controlling physical processes.

3) Apply the $\pi$ theorem [2] to the chosen parameters.

The $\pi$ theorem is a method for determining a dimensionally-correct relationship for a given set of parameters. A short description given by Binder [3] follows.

> Let $A_1A_2A_3 \ldots . A_n$ be n physical quantities which are involved in some physical phenomenon. Examples of these physical quantities are velocity, viscosity, and density. Let m be the number of all the primary or fundamental units (such as length, mass, and time) involved in this group of physical quantities. The physical equation, or the functional relation between these quantities, can be written as
>
> $$f(A_1\, A_2\, A_3 \ldots . A_m) = 0$$
>
> The $\pi$ theorem states that the foregoing relation can be written as
>
> $$\phi(\pi_1\, \pi_2 \ldots . \pi_{n-m}) = 0$$
>
> where each $\pi$ is an independent dimensionless product of some of the A's.

Thus, if there are n physical variables in a particular problem, and m fundamental units, then the physical relationship can be expressed in a form involving (n-m) dimensionless ratios. An elaborate and formal proof of the theorem is given by Buckingham [2]. A simplified step-by-step procedure and an example are given in appendix I.

As can be seen in appendix I, to equate these (n-m) ratios, the $\pi$ theorem introduces a dimensionless coefficient (P) which serves to make

the dimensionless ratios numerically equivalent. A practical example for the necessity of P is given in the second paragraph of section 5. The form of P is normally determined from actual test data. P can be a constant or it can be a function of some or all of the parameters used to determine the ratios. Buckingham [2] suggests several ways to determine the form of P.

One method is to let P be a function of all of the parameters of the process with the exception of the one for which the relationship is being developed. For ease of reference, this combination of parameters is called N. The $\pi$ theorem is entered with the chosen parameters and an expression for the dimensionless ratio N is obtained. Pairs of values of P and N, determined using observed data, are plotted together. Then the form of P(N) can be determined from this plot.

Once the dimensionless ratios and the form of P have been determined, then the results can be used to forecast the mixed-layer depth.

## 3. Variables and Processes that Affect the Mixed-Layer Depth During the Warm Season.

There is general agreement among many researchers that during the warming season the mixed-layer depth varies with:

1) wind (Geary [4], Tabata [5], Mazeika [6], Munk and Anderson [7]);

2) the heat flow between upper and lower layers (Mazeika [6], Rossby and Montgomery [8], Kitaigorodsky [1]);

3) coriolis effect (Rossby and Montgomery [8], Kitaigorodsky [1]);

4) divergence in the upper layers of the ocean (Mazeika [6], Tabata and Giovando [9]);

5) internal wave action (Mazeika [6], Tully [10], Tabata and Giovando [9]);

6) advection (Mazeika [6], Tully [10], Tabata and Giovando [9]).

Only the first three of these processes will be considered in developing a forecast method. The remainder of the influences contribute to the scatter of the results.

4. Selection of Parameters to Represent Controlling Processes.

An aid in determining representative parameters is to write the relevant dynamical equations describing the processes and to examine them term by term.

The equations of motion in the xy-plane, assuming an unaccelerated flow in an unbounded and horizontally-homogeneous ocean, reduce to:

$$
\begin{aligned}
0 &= f v + \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} \\
0 &= -f u + \frac{1}{\rho}\frac{\partial \tau_{zy}}{\partial z}
\end{aligned}
\tag{1}
$$

where u and v are the components of velocity, $\tau_{zx}$ and $\tau_{zy}$ are the components of stress on the horizontal plane, and $f$ is the coriolis parameter. At the interface, the stress is given by:

$$\tau = \rho \gamma^2 \overline{W}^2 \tag{2}$$

where $\tau$ is the stress on the water surface, $\overline{W}$ is the mean wind speed and $\gamma$ is a coefficient that is a function of wind speed and the height at which the wind is measured. This indicates that the water motions, including the eddies that carry both heat and momentum, depend upon the surface wind speed. Thus a measure of wind speed that is representative of the stress effects should be introduced as a parameter in the application of the $\Pi$ theorem. However, the depth to which wind effects reach depends also on the coriolis parameter as shown theoretically by Ekman [11]. Thus, the coriolis parameter must also be considered in application of $\Pi$ theory.

The maintenance of a thermal structure involves heat flow. A common equation for temperature changes due to heat flow is:

$$\rho c_p \frac{\partial T}{\partial t} = -\frac{\partial}{\partial z}\left(K \frac{\partial T}{\partial z}\right)$$

where $\rho$ is the density of water, $c_p$ is the specific heat of water, T is the temperature of the water, and K is the eddy-conductivity coefficient for heat. This indicates that the local rate of change of temperature is a function of K as well as of the differences in temperature between layers. But K depends upon the field of motion and the stability. Parameters to represent motion have already been chosen in the previous paragraph. Thus, to complete the representation of the variables in the heat flow process, the only additional parameters that need be introduced are those representative of stability.

Thus it appears that, for the conditions and assumptions given in section 3, the $\pi$ theorem should be entered with parameters that are representative of:

1) wind stress;

2) coriolis effect;

3) stability.

WIND STRESS. Here it is necessary to find a value of wind speed which is representative of the influence of wind on the MLD over a period of time. The wind enters into the mixed-layer-depth problem in the form of stress, and the stress coefficient ($\gamma$) is an increasing function of wind speed. Thus equations (1) and (2) show that higher winds should be weighted more heavily in estimating the influence of wind stress upon the MLD.

One therefore should avoid a linearly-averaged wind. The author has devised an empirical method which qualitatively and objectively

weights the stronger winds in order to obtain a suitably representative wind parameter. The parameter is called the "representative maximum wind" and is defined as:

> the average of the five highest winds of the eight usually reported during a chosen 24-hour period. The chosen period is that one having the highest winds in the interval from 72 hours to 12 hours prior to observation time.

The step-by-step procedure for determining the "representative maximum wind" is given in appendix III.

CORIOLIS EFFECT. The standard equation for the coriolis parameter times $10^4$ will be used to represent coriolis effects:

$$\Omega = f \times 10^4 = 2\,\omega \sin\phi \times 10^4$$

The constant, $1 \times 10^4$, was introduced arbitrarily for convenience.

STABILITY. The density gradient, and thus the stability, can be represented by a combination of the coefficient of thermal expansion ( $\beta$ ) and the temperature difference between layers; the following two parameters have been chosen to represent stability:

1) the coefficient of thermal expansion ($\beta$);

2) the excess heat (Q) in the mixed layer over that in lower layers.

Values of $\beta$ used in this paper were interpolated from values given by Sverdrup [12] assuming a constant salinity of 32.5 ‰. These values are tabulated in appendix III. Constant salinity was assumed at OWS Papa during the warm season as in Tabata [5].

The step-by-step methods used to determine Q are given in appendix III. One difficulty with the use of Q as a measure of stability is illustrated in figure 2. This shows that thermal structures of obviously

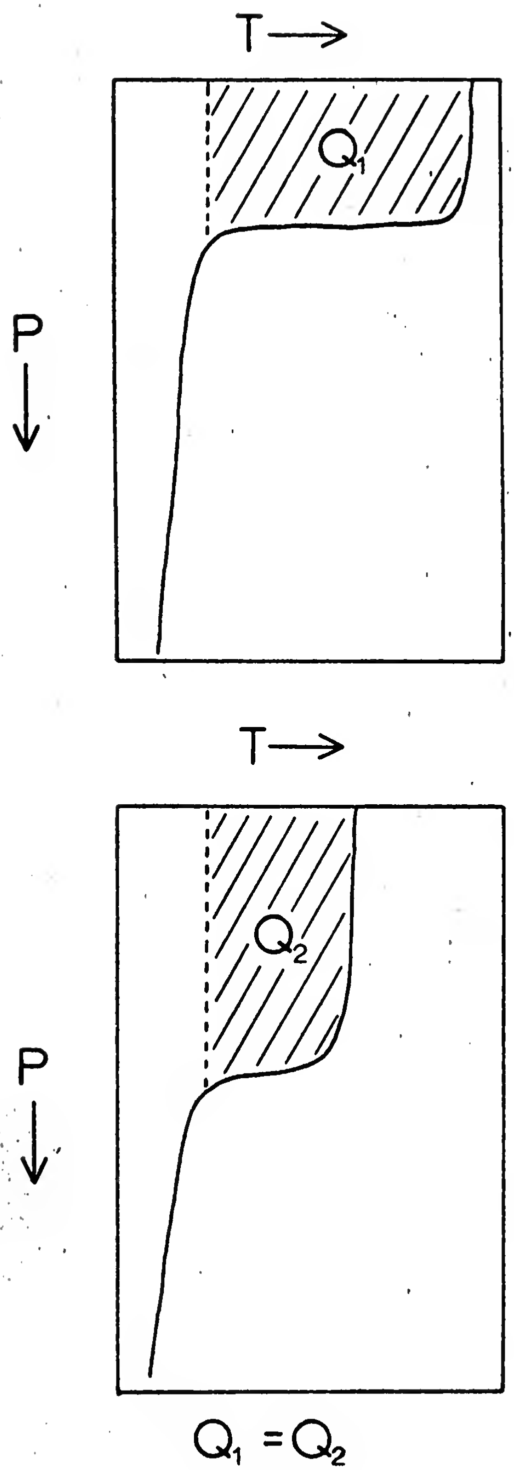


Figure 2

Two Possible Thermal Structures with
Same Excess Heat Present in Warm Layer

different stability can prevail with the same Q. However, the combination of Q, $\beta$, and the MLD give a good approximation of the stability of the mixed layer. (The MLD enters the forecast model in the form of P(N) since P(N) is initially determined from raw data that includes MLDs.)

Thus, besides the MLD itself, four parameters have been chosen for entry into the $\pi'$ theorem:

1) the representative maximum wind (W);

2) the coriolis parameter ($f$);

3) the coefficient of thermal expansion ($\beta$);

4) the excess heat in the upper layer (Q).

5. Practical Application of Similarity Theory.

In section 4, four parameters were chosen as representative of the processes effective in forming and maintaining a mixed layer at OWS Papa in the warm season. When the $\pi$ theorem is applied to these four parameters the results are:

$$MLD = P(N)\frac{W^2}{Q\beta\Omega^2} \tag{3}$$

$$N = \frac{Q\beta\Omega}{W} \tag{4}$$

Both of these equations are derived in the example in appendix I.

Because the relationship between MLD and the other parameters is complex, the coefficient P(N) is not in general a constant and must be found as a function of some of the parameters, as in appendix I. As an example of the ambiguity which must be eliminated by use of P, consider again the parameter Q; as shown in fig. 2, two quite different thermal structures may have the same excess heat Q. Thus a value of Q alone does not uniquely specify the thermal structure. Rather the structure is a result of the interaction of all the controlling parameters. That is, a wide range of possible mixed-layer depths exists for each value of a particular controlling parameter, whereas only one MLD exists for each combination of parameters. To specify the form of P(N), which is a function of all the prevailing parameters, several pairs of values of P and N are determined from experimental data and plotted together. Then the form of P(N) is given by this plot.

If, in the application of similarity theory, parameters truly representative of the controlling processes are chosen, then the plot of P

versus N should have little scatter. If the parameters used are not truly representative, then large scatter results.

To test equations (3) and (4) they were used to obtain paired values of P and N; these values were based on mean monthly data at OWS Papa tabulated by Tabata and Giovando [9]. Values of the parameters, W, $\beta$ , $\Omega$, and $Q_S$ (where the subscript S refers to seasonal thermoclines) were determined in accordance with Appendix III. The paired values of P and N are plotted in figure 3. The points have very little scatter; and, in fact, a straight line gives a very good least-squares fit. However, the use of figure 3 is limited because it represents a climatological MLD, whereas the MLD for a particular day can vary considerably from the climatological MLD. Figure 3 is important in that the small scatter of the paired values apparently indicates that the proper parameters have been used in equations (3) and (4).

In order to make equations (3) and (4) useful in short range prediction of the MLD, P(N) was determined using daily data from OWS Papa. The data were from June to October during the years 1958 through 1962. In order to filter out internal wave effects, data were used only from those days for which six or more bathythermographs were available; the MLD used is a mean of the six (or more) bathythermographs. This requirement limited the available data such that only 22 pairs of P and N are available for transitional MLDs and 29 pairs for seasonal MLDs. Although the number of paired values appears small, they represent over 200 BTs.

Pairs of values of P and N were first obtained for MLDs associated with transitional thermoclines with values of the parameters W, $\beta$ , $\Omega$, and $Q_T$ determined as stated in Appendix III. Note that for a

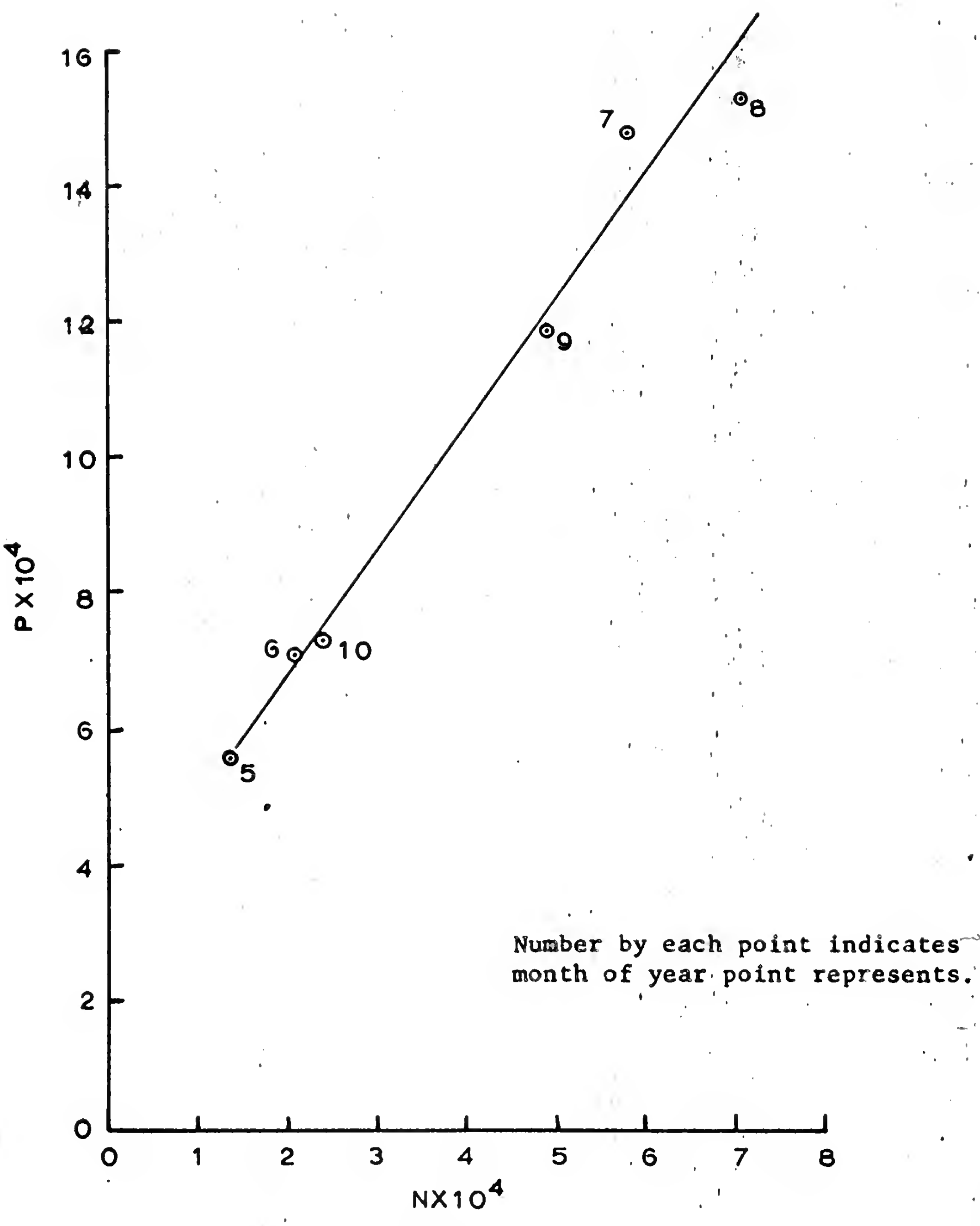


Figure 3

P versus N Based on Monthly Climatological Data
for the Seasonal Thermocline at OWS Papa

transitional thermocline the general parameter Q in equations (3) and (4) is represented by the more precise parameter $Q_T$. Figure 4 is a plot of pairs of P and N resulting from data listed in table 1. The least-squares best-fit straight line has the equation:

$$P = -.25 \times 10^{-4} + 2.9\,N$$

Substituting this into equation (3) gives:

$$MLD = -.25 \times 10^{-4}\left(\frac{W^2}{Q_T\,\beta\,\Omega^2}\right) + 2.9\left(\frac{W}{\Omega}\right) \tag{5}$$

Thus equation (5) can be used to forecast a transitional MLD without determining a value of P.

It is interesting to note the similarity between equation (5) and the equation derived by Rossby and Montgomery [8]:

$$MLD = -K_1 W_o^2 + K_2 \frac{W_o}{\sin\phi} \tag{6}$$

$W_o$ = wind speed measured at the sea surface.

$\phi$ = latitude

Based on 39 observations, they arrived at a value for $K_2$ of 2.38 per second. It was concluded that $K_1$ was negligible. The MLD data for these 39 observations were obtained by interpolating between readings of temperature and salinity which are 10-25 meters apart. The average MLD for the 39 observations was 32 meters, and they considered a five-meter average error as a plausible assumption. Since the data used by Rossby and Montgomery appeared to be of poor quality compared to the data available at OWS Papa, it is suggested that equation (5) is more applicable than (6) for forecasting the transitional thermocline. Tests in section 6

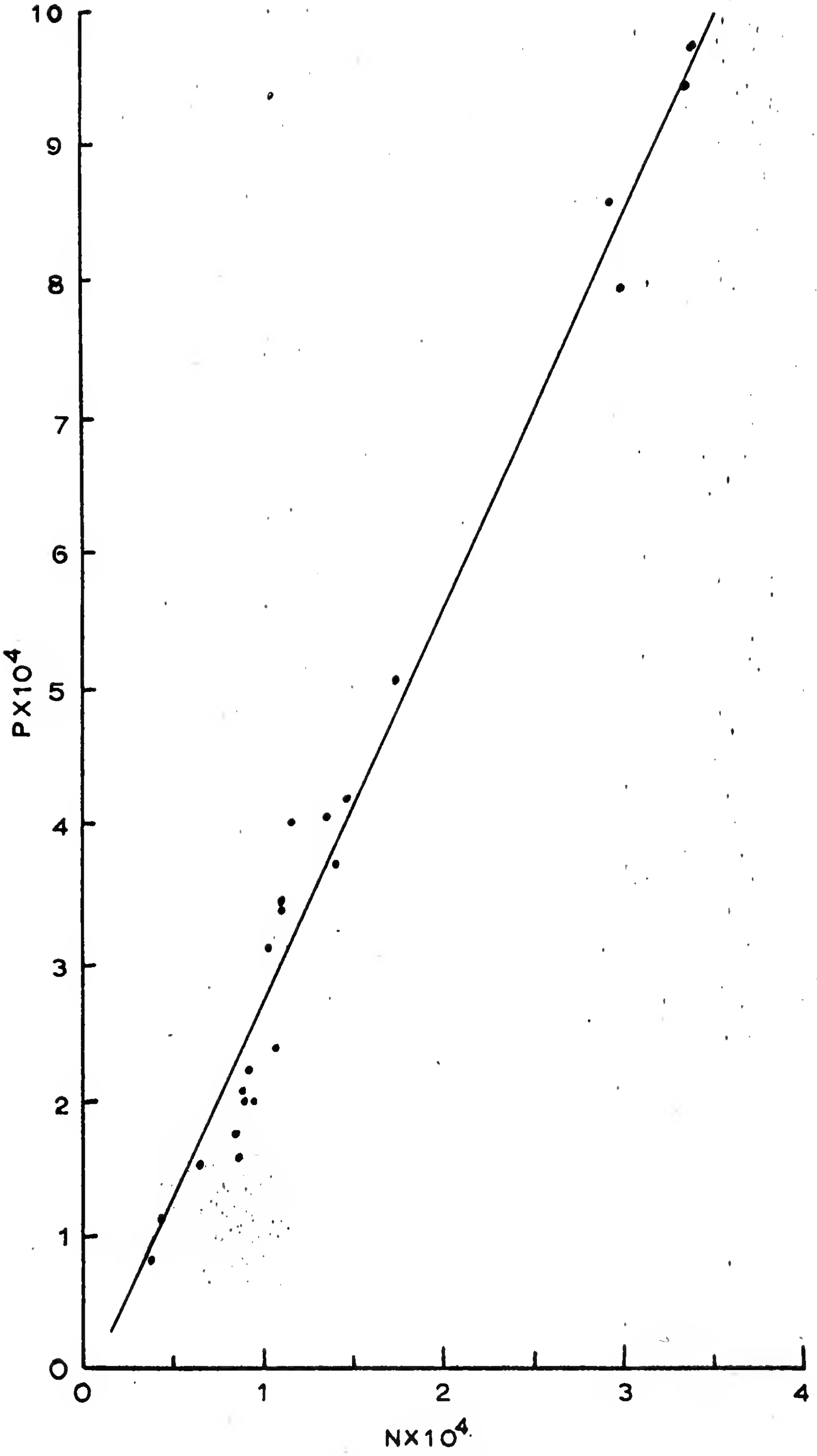


Figure 4
P versus N for OWS Papa Data (Transitional Thermocline) with Least-Squares Best-Fit Line

TABLE 1

DATA USED TO DETERMINE VALUES OF P AND N
FOR MIXED-LAYER DEPTHS ASSOCIATED WITH TRANSITIONAL THERMOCLINES

| DATE | W (KNOTS) | $Q_T$ (Kg cal/cm$^2$) | MLD (METERS) | TS ($^oC$) | P x $10^4$ | N x $10^4$ |
|---|---|---|---|---|---|---|
| 61758 | 27.4 | 7.21 | 26.4 | 10.3 | 2.0 | 0.96 |
| 61858 | 27.4 | 6.39 | 23.4 | 10.3 | 1.6 | 0.85 |
| 62058 | 27.8 | 8.06 | 28.5 | 10.4 | 2.4 | 1.10 |
| 71758 | 20.0 | 3.78 | 20.4 | 13.2 | 1.8 | 0.80 |
| 81658 | 25.6 | 18.29 | 34.1 | 12.5 | 8.6 | 2.95 |
| 71062 | 20.0 | 9.00 | 27.2 | 10.7 | 4.1 | 1.30 |
| 71262 | 21.0 | 7.25 | 30.0 | 10.8 | 5.0 | 1.71 |
| 80661 | 19.6 | 13.50 | 23.9 | 13.5 | 8.0 | 3.00 |
| 90661 | 26.6 | 21.40 | 34.8 | 13.3 | 9.7 | 3.40 |
| 91461 | 30.6 | 24.00 | 39.7 | 13.4 | 9.4 | 3.33 |
| 62362 | 12.3 | 2.98 | 13.9 | 10.5 | 2.3 | 0.92 |
| 62762 | 24.0 | 5.86 | 27.3 | 9.5 | 2.1 | 0.86 |
| 63062 | 24.0 | 4.01 | 27.4 | 10.0 | 1.5 | 6.10 |
| 70262 | 20.0 | 6.32 | 32.4 | 10.1 | 4.1 | 1.15 |
| 70662 | 25.0 | 9.63 | 30.3 | 10.4 | 3.7 | 1.40 |
| 70762 | 25.0 | 10.11 | 32.4 | 10.3 | 4.2 | 1.47 |
| 71462 | 20.0 | 5.38 | 27.6 | 11.0 | 3.1 | 1.04 |
| 71662 | 20.0 | 5.87 | 28.0 | 11.2 | 3.5 | 1.13 |
| 72062 | 17.2 | 4.86 | 24.3 | 11.6 | 3.4 | 1.12 |
| 72662 | 13.2 | 1.10 | 14.1 | 13.0 | 0.8 | 0.35 |
| 72862 | 17.0 | 1.83 | 19.8 | 13.0 | 1.2 | 0.46 |
| 73062 | 22.8 | 5.34 | 23.0 | 12.7 | 2.1 | 0.96 |

indicate this to be true and also show that the terms of equation (5) are all of the same order of magnitude.

Although equation (5) apparently is applicable for MLDs associated with the transitional thermocline, tests indicated that (5) was no longer applicable when the transitional MLD joins the seasonal MLD. In this case, a slightly different approach is necessary, apparently due to the increased stability associated with a seasonal type thermocline. To take this increased stability into consideration $Q_S$ (defined in Appendix III) is used in equations (3) and (4) instead of $Q_T$.

The general procedure for finding values of P and N for the seasonal MLD was the same as for values associated with the transitional MLD with the exception that $Q_S$ was used instead of $Q_T$. Pairs of values of P and N for seasonal MLDs, based on data listed in table 2, are plotted in figure 5. The least-squares best-fit line for these points was:

$$P = -6.1 \times 10^{-4} + 3.89\, N$$

Substituting into equation (3) gives:

$$MLD = -6.1 \times 10^{-4} \left( \frac{W^2}{Q \beta \Omega^2} \right) + 3.89 \left( \frac{W}{\Omega} \right) \tag{7}$$

Equation (7) is applicable to MLDs associated with the seasonal thermocline.

An objective method has been devised to determine whether a transitional or seasonal condition exists. When $N < 3.5 \times 10^{-4}$, the thermocline is defined as transitional. When $N > 3.5 \times 10^{-4}$, a seasonal thermocline is said to exist. This is not a rigid rule. Obviously, even when $N < 3.5 \times 10^{-4}$, if the forecast MLD using equation (5) is deeper than a prevailing seasonal thermocline, then equation (7) should be used rather than (5).

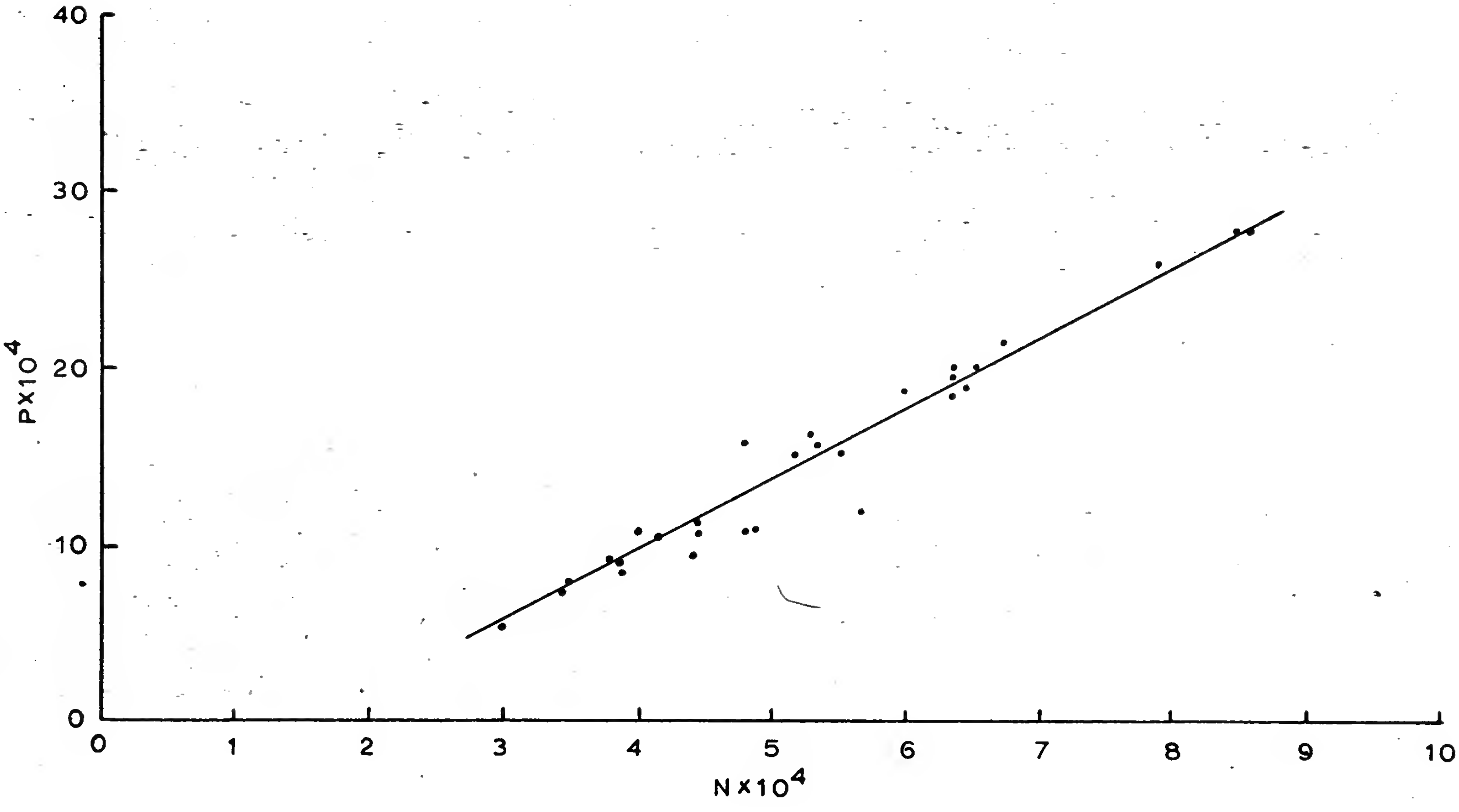


Figure 5

P versus N for OWS Papa Data (Seasonal Thermocline)
with Least-Squares Best-Fit Line

TABLE 2

DATA USED TO DETERMINE VALUES OF P AND N FOR MIXED-LAYER DEPTHS ASSOCIATED WITH SEASONAL THERMOCLINES

| DATE | W (KNOTS) | Qs (Kg cal/cm$^2$) | MLD (METERS) | TS (°C) | P x $10^4$ | N x $10^4$ |
|---|---|---|---|---|---|---|
| 62762 | 24.0 | 28.3 | 27.3 | 9.5 | 10.3 | 4. |
| 91958 | 50.0 | 44.1 | 49.8 | 11.6 | 7.6 | 3.5 |
| 61858 | 27.4 | 22.5 | 23.4 | 10.3 | 5.6 | 3.0 |
| 91161 | 27.2 | 43.2 | 40.5 | 13.3 | 21.8 | 6.7 |
| 82761 | 21.4 | 40.6 | 32.2 | 14.5 | 27.7 | 8.5 |
| 82361 | 21.0 | 41.4 | 31.3 | 13.8 | 27.7 | 8.6 |
| 80661 | 19.6 | 35.7 | 23.9 | 13.5 | 21.0 | 7.9 |
| 72062 | 17.2 | 28.5 | 24.3 | 11.6 | 20.2 | 6.6 |
| 90661 | 26.6 | 39.7 | 34.8 | 13.3 | 18.0 | 6.3 |
| 91461 | 27.2 | 40.6 | 39.7 | 13.4 | 20.1 | 6.3 |
| 73062 | 23.0 | 31.8 | 22.8 | 12.7 | 12.3 | 5.7 |
| 90961 | 27.2 | 40.7 | 37.6 | 13.4 | 19.1 | 6.3 |
| 71462 | 20.0 | 31.0 | 27.6 | 11.0 | 18.0 | 6.0 |
| 71662 | 20.0 | 27.5 | 28.0 | 11.2 | 16.2 | 5.3 |
| 91461 | 30.6 | 39.5 | 39.7 | 13.4 | 15.5 | 5.5 |
| 81261 | 28.4 | 32.5 | 30.3 | 13.2 | 11.3 | 4.9 |
| 71262 | 21.0 | 29.6 | 30.0 | 10.7 | 16.6 | 5.3 |
| 71062 | 20.0 | 27.5 | 27.2 | 10.8 | 15.4 | 5.2 |
| 70262 | 20.0 | 26.2 | 32.4 | 10.1 | 16.9 | 4.8 |
| 80861 | 28.0 | 29.6 | 26.8 | 13.3 | 9.3 | 4.5 |
| 81461 | 32.0 | 37.4 | 34.7 | 12.8 | 11.4 | 4.8 |
| 63062 | 24.0 | 29.1 | 27.4 | 10.0 | 11.0 | 4.4 |
| 70662 | 25.0 | 30.5 | 30.3 | 10.4 | 11.7 | 4.5 |
| 70762 | 25.0 | 27.6 | 32.4 | 10.3 | 11.4 | 4.0 |
| 61758 | 27.4 | 29.5 | 26.4 | 10.4 | 8.2 | 3.9 |
| 61958 | 27.4 | 29.8 | 25.5 | 10.4 | 8.0 | 4.0 |
| 62058 | 27.8 | 29.3 | 28.5 | 10.3 | 8.6 | 3.8 |
| 81658 | 25.6 | 39.8 | 34.1 | 12.5 | 18.6 | 6.4 |
| 91858 | 50.0 | 43.7 | 49.4 | 11.5 | 7.4 | 3.4 |

6. Application and Test of Equations.

Equations (5) and (7) can be used to forecast MLD over any length of time for which the parameters can be accurately predicted. Thirteen tests of the equations were made using independent data for forecast periods of from one to four days. Five of the tests were based on data from OWS Papa (50N, 145W) and eight on data from OWS November (30N, 140W). Three of the tests were for transitional thermoclines and ten were for seasonal thermoclines. The tests were made using a mean MLD computed from four to six BTs from each day as a representative MLD for that day. Results are shown in table 3.

The verification of a forecast is made difficult by the fact that the MLD varies randomly. One objective way to measure verification success is to determine if the forecast value lies within the normal range of variability of the MLD as measured by the standard deviations of the individual values. In table 3 the 13 values listed under "verified MLD" actually represent 63 BTs. The standard deviation with respect to the daily mean for these 63 bathythermographs was 3.47 meters. The percentage of forecasts that fell within one and two standard deviations of the individual MLDs is given in table 4. The root-mean square and the algebraic mean of the forecast differences are also given in table 4.

The algebraic mean of the forecast was computed to determine if the forecasts were biased. Forecasts using the equation proposed by Rossby and Montgomery were biased positively whereas those based on equations (5) and (7) were less biased and negative. This suggests that the negative term neglected by Rossby and Montgomery (see section 5) should be included as in equations (5) and (7).

TABLE 3

FORECAST VERIFICATIONS

| LOCATION OWS | INITIAL DATE | INITIAL MLD | FINAL DATE | VERIFIED MLD | FCST MLD BY EQUATIONS 5&7 (FCST DIFFERENCE) | | FCST MLD BY PERSISTENCE (FCST DIFFERENCE) | | FCST MLD BY ROSSBY & MONTGOMERY (FCST DIFFERENCE) | |
|---|---|---|---|---|---|---|---|---|---|---|
| P | 72859 | 30.0 | 80159 | 34. (T) | 35.0 | (+1) | 30.0 | (-4) | 50.5 | (+16.5) |
| P | 71959 | 33.0 | 72159 | 35. (T) | 30.0 | (-5) | 33.0 | (-2) | 39.0 | (+5.0) |
| P | 81757 | 21.0 | 81957 | 9.4(T) | 6.8 | (-2.7) | 21.0 | (+11.6) | 10.0 | (+0.6) |
| P | 81957 | 9.4 | 82357 | 18.3(S) | 16.1 | (-2.2) | 9.4 | (-8.9) | 24.1 | (+5.8) |
| P | 82357 | 18.3 | 82657 | 16. (S) | 15.0 | (-1) | 18.3 | (+2.3) | 23.0 | (+7.0) |
| N | 70657 | 36.0 | 70757 | 43. (S) | 37.1 | (-5.9) | 36.0 | (-7.0) | 48.0 | (+5.0) |
| N | 70757 | 43.0 | 71057 | 37.5(S) | 35.3 | (-2.2) | 43.0 | (+5.5) | 54.1 | (+5.5) |
| N | 71957 | 42.0 | 72157 | 44.5(S) | 41.0 | (-3.5) | 42.0 | (-2.5) | 61.0 | (+16.5) |
| N | 72457 | 40.5 | 72757 | 49. (S) | 37.0 | (-12.0) | 40.5 | (-8.5) | 50.0 | (+1.0) |
| N | 61457 | 15.0 | 61657 | 27.9(S) | 30.0 | (+2.1) | 15.0 | (-12.9) | 37.0 | (+8.1) |
| N | 82458 | 38.1 | 82558 | 34.7(S) | 35.2 | (+0.5) | 38.1 | (+3.4) | 42.5 | (+7.8) |
| N | 72959 | 46.0 | 73159 | 43.5(S) | 39.3 | (-4.2) | 46.0 | (+2.5) | 52.0 | (+8.5) |
| N | 62959 | 33.0 | 70159 | 32. (S) | 29.5 | (-2.5) | 33.0 | (+1.0) | 50.5 | (+18.5) |

(T) indicates a transitional thermocline.
(S) indicates a seasonal thermocline.

TABLE 4

STATISTICAL RESULTS OF FORECASTS

| | FORECASTS BY EQUATIONS (5) AND (7) | FORECASTS BY PERSISTENCE | FORECASTS BY ROSSBY AND MONTGOMERY EQUATION |
|---|---|---|---|
| % FORECASTS WITHIN 1 STANDARD DEVIATION | 69 | 46 | 15 |
| % FORECASTS WITHIN 2 STANDARD DEVIATIONS | 92 | 61.5 | 46 |
| ROOT MEAN SQUARE OF DIFFERENCES (METERS) | 4.3 | 6.3 | 11.1 |
| ALGEBRAIC MEAN OF DIFFERENCES (METERS) | -2.9 | -1.06 | 8.1 |

To test how the function P(N) at OWS November compares with that found at OWS Papa, eight paired values of P and N at OWS November were computed. The BTs upon which the verifications were based in table 3 were also used to obtain these paired values in the same manner as for OWS Papa (section 5). Figure 6 is a plot of paired values of P and N for the data listed in table 5; the straight line represents P(N) for the constants derived from data at OWS Papa. The function P(N) appears to be very nearly the same at both OWS Papa and OWS November.

Two examples of the application of equations (5) and (7) to forecasting follow.

Example 1. Forecast to be made for 23 August 1957 based on BTs taken on 19 August at OWS Papa.

1. From the BTs taken on 19 August 1957, $Q_T$ was determined to be 1.74 Kg cal/cm$^2$ by methods described in Appendix III.

2. Using the August climatological data computed by Tabata and Giovando [9], the average net heat transfer downward across the air-sea boundary at OWS Papa is approximately +0.1 Kg cal/cm$^2$ per day. Using this information, 0.4 Kg cal/cm$^2$ were added to $Q_T$ during the four-day interval up to the verification day giving:

$$Q_T = 1.74 + 0.4 = 2.14 \text{ Kg cal/cm}^2$$

(Where enough current meteorological data are available, the methods for computing the net heat transfer proposed by Laevastu [13] could be applied instead of climatology.)

3. Values of $\beta$, $\Omega$, and W were determined by the methods described in Appendix III.

4. Equation (4) was entered and N was found to be less than

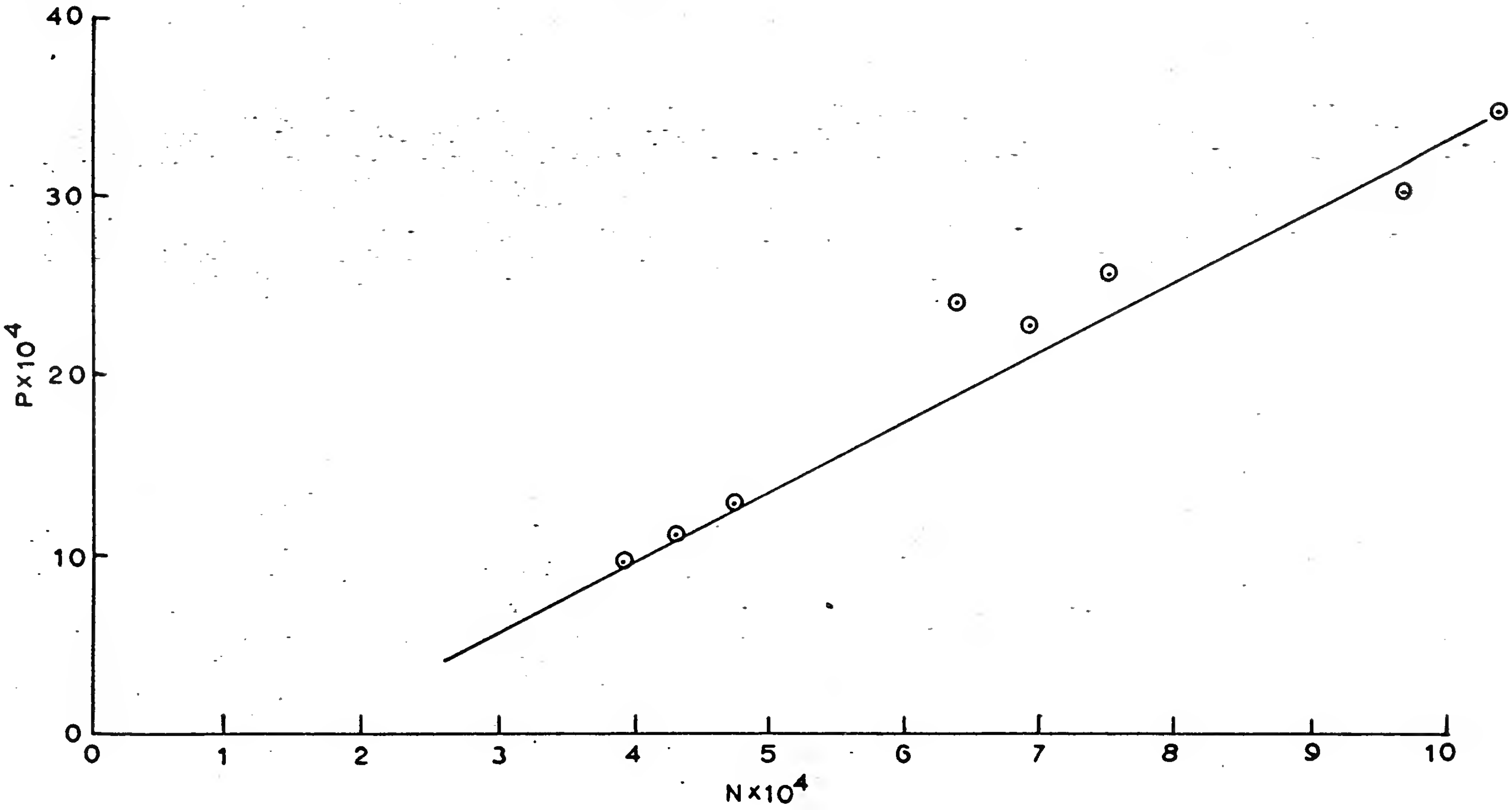


Figure 6

P versus N for OWS November Data
Compared to OWS Papa P versus N Best-Fit Line (Seasonal Thermocline)

TABLE 5

DATA USED TO DETERMINE VALUES OF P AND N FOR MIXED LAYER DEPTHS ASSOCIATED WITH SEASONAL THERMOCLINES AT OWS NOVEMBER

| DATE | MLD (METERS) | W (KTS) | $Q_S$ (Kg cal/cm$^2$) | P x $10^4$ | N x $10^4$ |
|---|---|---|---|---|---|
| 61657 | 27.9 | 13.2 | 32.9 | 26.8 | 7.5 |
| 70757 | 43.0 | 17.1 | 31.6 | 13.0 | 4.7 |
| 71057 | 37.5 | 19.3 | 22.7 | 11.5 | 4.3 |
| 72157 | 44.5 | 23.5 | 25.2 | 24.9 | 6.4 |
| 72757 | 49.0 | 17.8 | 28.3 | 28.5 | 9.7 |
| 82558 | 34.7 | 15.1 | 39.6 | 34.2 | 10.5 |
| 73159 | 43.5 | 18.5 | 33.0 | 23.1 | 6.9 |
| 70159 | 32.0 | 18.0 | 18.2 | 9.9 | 3.9 |

$3.5 \times 10^{-4}$. Consequently, equation (5) was used, giving a forecast MLD of 16.1 meters.

5. The seasonal MLD was at 32 meters on 19 August 1957. According to the above forecast, the wind was not strong enough to cause the transitional thermocline to merge with the seasonal thermocline. Thus persistence is used to forecast the depth of this seasonal thermocline.

6. The forecast thermal structure for 23 August 1957 consists of a mixed layer down to a transitional thermocline at 16.1 meters with a seasonal thermocline below it at 32 meters. The verification showed a transitional thermocline at 18.3 meters and a seasonal thermocline at 30.5 meters.

Example 2. For the forecast MLD of 1 August 1959 at OWS Papa, steps 1, 2 and 3 as done in example 1 resulted in $N > 3.5 \times 10^{-4}$. Consequently, the procedure for a seasonal thermocline was followed. That is, $Q_S$ was determined and equation (7) was used to forecast the MLD. The forecast MLD was 35 meters and 39 meters verified.

## 7. Conclusions.

The agreement between forecast and observed MLDs tends to verify the findings of Kitaigorodsky that similarity theory is useful in this forecasting problem. Equations (5) and (7) were successful in forecasting both large and small changes in the MLD, in the positive as well as in the negative sense. The concept of a universal function P(N), as proposed by Kitaigorodsky, is strengthened.

Equations (5) and (7) appear to be useful in forecasting the mixed layer depth to a reasonable degree of accuracy. Some of the deviations can be attributed to the effects of divergence, advection and internal waves which were not evaluated in this study.

Future research could well be applied to determining more paired values of P and N for other ocean locations. This would not only help to fix the constants of equations (5) and (7) better, but would help to demonstrate further the universality of P(N), if it exists.

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APPENDIX I

BASIC PROCEDURES FOR APPLICATION OF $\pi$ THEOREM

Let n represent the number of parameters ($A_1\ A_2 \ldots A_n$) chosen to be representative of the processes involved in a particular physical phenomenon. Let m represent the number of fundamental dimensions (length, time, mass etc.) present in all of the parameters. Then follow these steps.

1. Determine the number of " $\pi$ dimensionless ratios" required to incorporate all the chosen parameters (given by n-m).

2. Combine m + 1 parameters into an equation of the following form:

$$\pi_1 = A_1^{x_1} A_2^{y_1} A_3^{z_1} \ldots\ldots A_{m+1} \tag{8}$$

where ($A_1\ A_2 \ldots A_m$) must together include all of the fundamental dimensions.

3. Substitute the fundamental dimensions into equation (8) and equate like powers on each of the fundamental dimensions to zero,

$$A_1^{x_1} A_2^{y_1} \ldots\ldots A_{m+1} = L^0 M^0 T^0 t^0$$

4. Solve the resulting m equations with m unknowns to obtain $x_1$, $y_1$ and $z_1$.

5. Substitute the values of $x_1, y_1, z_1$ into the equation for $\pi_1$ to obtain the first dimensionless ratio.

6. Repeat steps 2 through 5 after changing $A_{m+1}$ to another parameter not previously used for any A and solve for $\pi_2$. Repeat as necessary until $\pi_1$, $\pi_2$, ....$\pi_{n-m}$ ratios are obtained.

7. The ratios represented by $\pi_1$, $\pi_2$, ....$\pi_{n-m}$ are all dimensionless

and can be equated one to the other to obtain a functional relationship that is dimensionally correct. The ratios are then made numerically equivalent by use of a dimensionless coefficient P.

An example illustrates the application of the foregoing step by step procedure.

| | Parameters | Dimensions |
| --- | --- | --- |
| Q | (total heat present) | LT |
| W | (wind speed) | L/t |
| $\Omega$ | (coriolis effect) | 1/t |
| $\beta$ | (coefficient of thermal expansion) | 1/T |
| H | (mixed-layer depth or MLD) | L |
| | n = 5 | m = 3 |

STEP 1 n-m = 2 " $\pi$ ratios" will be required

STEP 2 $\pi_1 = Q^{x_1} W^{y_1} \Omega^{z_1} \beta$ and $\pi_2 = Q^{x_2} W^{y_2} \Omega^{z_2} H$

To find the powers (x,y,z) which make the above products dimensionless:

STEP 3

For $\pi_1$: $(LT)^{x_1} (L/t)^{y_1} (1/t)^{z_1} (1/T) = L^0 T^0 t^0$

STEP 4

(L) $x_1 + y_1 = 0$

(T) $x_1 - 1 = 0$

(t) $-y_1 - z_1 = 0$

$$x_1 = 1$$

$$y_1 = -1$$

$$z_1 = 1$$

STEP 5

$$\Pi_1 = Q^1 W^{-1} \Omega^1 \beta$$

or

$$\Pi_1 = \frac{Q \Omega \beta}{W}$$

STEP 6 For $\Pi_2$: $(LT)^{x_2} (L/t)^{y_2} (1/t)^{z_2} (L) = L^0 T^0 t^0$

(L) $x_2 + y_2 + 1 = 0$

(T) $x_2 = 0$

(t) $-y_2 - z_2 = 0$

$$x_2 = 0$$

$$y_2 = -1$$

$$z_2 = 1$$

$$\Pi_2 = Q^0 W^{-1} \Omega^1 H$$

$$\Pi_2 = \frac{\Omega H}{W}$$

STEP 7 Since both $\Pi_1$ and $\Pi_2$ are dimensionless, either can be inverted in order to give a relation that agrees with what is observed physically. In this case it is known that the MLD is an increasing function of the wind, and $\Pi_1$ and $\Pi_2$ may be combined into:

$$\Pi_2 = P \frac{1}{\Pi_1}$$

where P is a dimensionless coefficient whose form must be determined experimentally. Substituting for $\Pi_1$ and $\Pi_2$ and solving for the mixed layer depth gives:

$$MLD = P \left( \frac{W^2}{Q \beta \Omega^2} \right)$$

In his paper on $\pi$ theory, Buckingham [2] gave several methods to determine the form of P. Of these, the same method used by Kitaigorodsky [1] will be given as an example here. Since P may be considered a function of all the original parameters, they can be used to determine P with experimental data. If P does not turn out to be constant, it can be made a function of certain of the parameters and the form of that function can be found from the data. Thus P is made a function of N, which, in this example, is made to depend upon all the parameters except MLD, the quantity to be predicted. Use of $\pi$ theory, as before, yields a dimensionless ratio of these parameters:

$$N = \frac{Q \beta \Omega}{W}$$

Entering the equations for both P and N with the same data and plotting the corresponding pairs gives P(N) in graphical form.

# APPENDIX II

## RELATIONSHIP BETWEEN VARIANCE OF THE MLD AND THE STRENGTH OF THE THERMOCLINE

Mazeika [6] used 356 groups of BT data to determine that the MLD at OWS Echo had fluctuations with a mean amplitude of 17.85 feet and standard deviation of 10.55 feet about the mean. He attributed these fluctuations to internal waves.

On the basis of 100 BT observations from OWS Papa for the months of September and October of the years 1958 through 1960, the author determined a system for estimating the amount of fluctuation of the MLD that might be expected at verification time. The temperature difference between the sea surface and the temperature of the near-isothermal water just below the thermocline was used as a measure of the strength of the thermocline. The variance of each observation was computed from a five-day running mean. The average variance and the average $\Delta T$ for each two-meter increment of mixed layer depth was then computed and plotted as pairs in figure 7. Figure 7 can be used to estimate the variance that can be expected at verification time.

In order to smooth out the fluctuations of the MLD when an accurate verification is desired, it is apparently best to take a series of BT observations over a period of at least 12 hours and then use the average of all the BTs to verify the forecast.

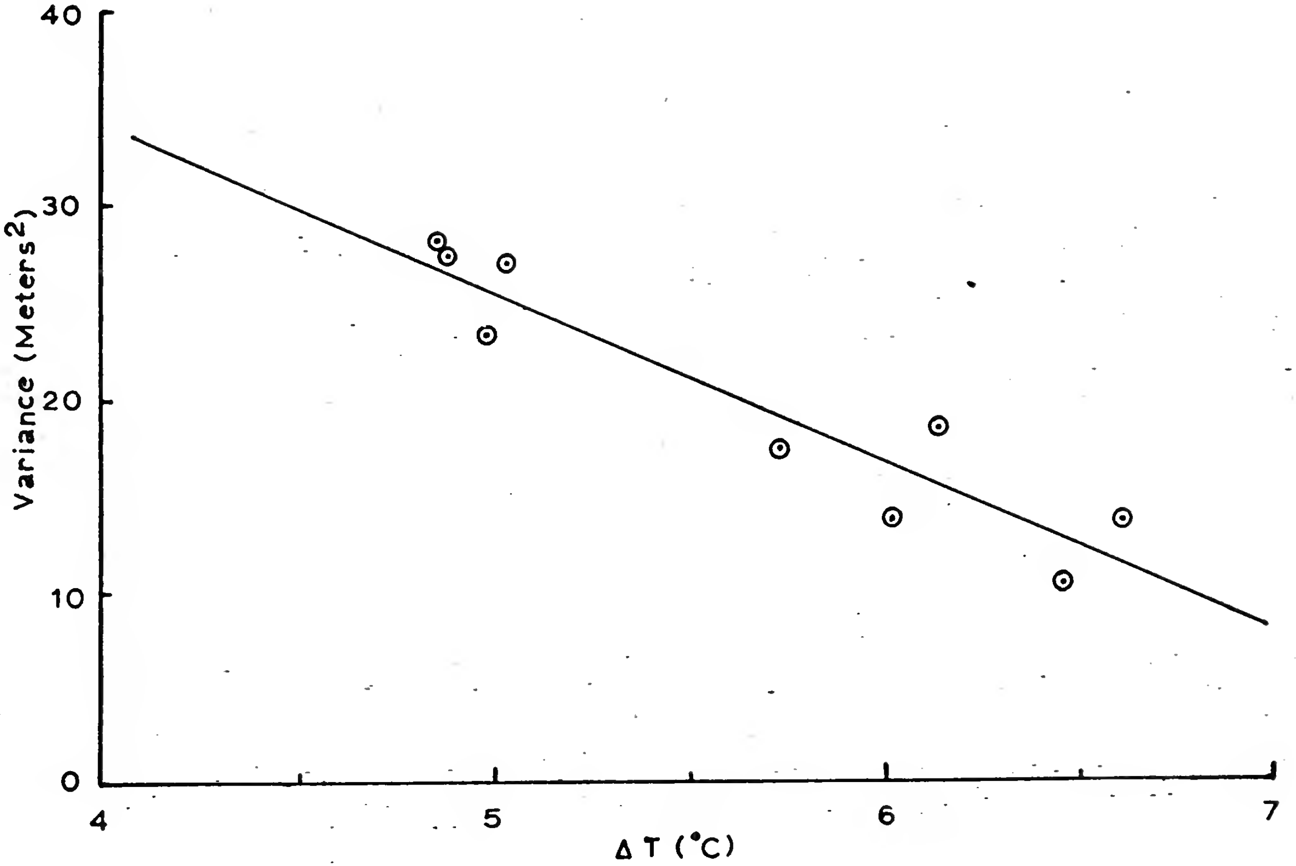


Figure 7

Variance of MLD about the Mean versus Temperature Change
Through the Uppermost Thermocline

## APPENDIX III

## METHODS USED FOR DETERMINING VALUES OF PARAMETERS

### WIND (W)

1. Out of the 72-hour period preceding verification time, but not including the 12 hours immediately preceding verification time, determine the 24-hour period during which the highest average winds prevailed or are forecasted to occur.

2. From the eight wind reports given for that 24-hour period, choose the five highest speeds. The average of these five is the "representative maximum wind."

### EXCESS HEAT IN UPPERMOST LAYER (Q)

When finding values of P and N, the computation of Q was based on the average BT for each particular 12-hour period. When using equations (5) and (7) to forecast an MLD, the most recent BTs available should be used to compute Q. In either case the following steps are applicable in determining the excess heat in the uppermost layer.

1. Assume a transitional warm layer exists. For this situation compute $Q_T$ as follows:

   a) Determine "$AREA_T$", depicted in figure 8. The dashed line is a vertical drawn from the point of maximum curvature of the BT trace (roughly the bottom of the thermocline) to the surface.

   b) $Q_T$ is then given by

$$Q_T = \rho c_p AREA_T \times 10^{-1} \qquad (Kg\ cal/cm^2)$$

For OWS Papa in summer $\rho c_p \cong 0.975$ for salinity 32.5 $^{\circ}/_{oo}$, assumed to be constant ($\rho c_p$ in $\frac{cal}{^{\circ}C\ cm^3}$ and $AREA_T$ in $M^{\circ}C$).

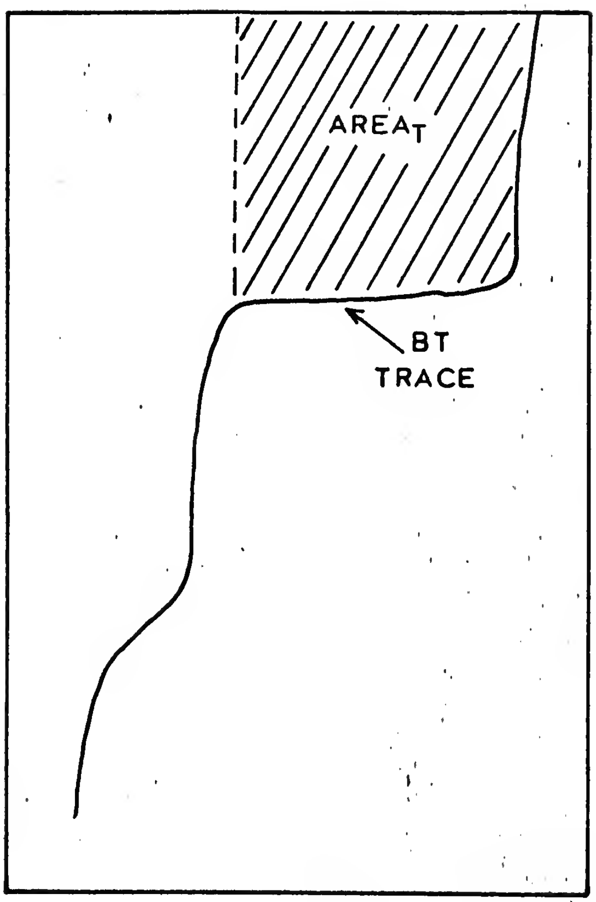


Figure 8

Representation of Area$_T$
Used in Determining $Q_T$

2. If $N > 3.5 \times 10^{-4}$ using $Q_T$, then a seasonal-type thermocline is assumed to exist. The excess heat for a seasonal thermocline ($Q_S$) is found by the following objective method.

a) Determine "$AREA_S$", depicted in figure 9(d). The dotted line, figure 9(a), is a vertical drawn from the intersection of the BT trace and 200 meters. HH is determined by equalizing areas A and B as in figure 9(b). Then

$$AREA_S = C - B$$

b)

$$Q_S = \rho c_p \, AREA_S \times 10^{-1} \qquad (\mathrm{Kg\,cal/cm^2})$$

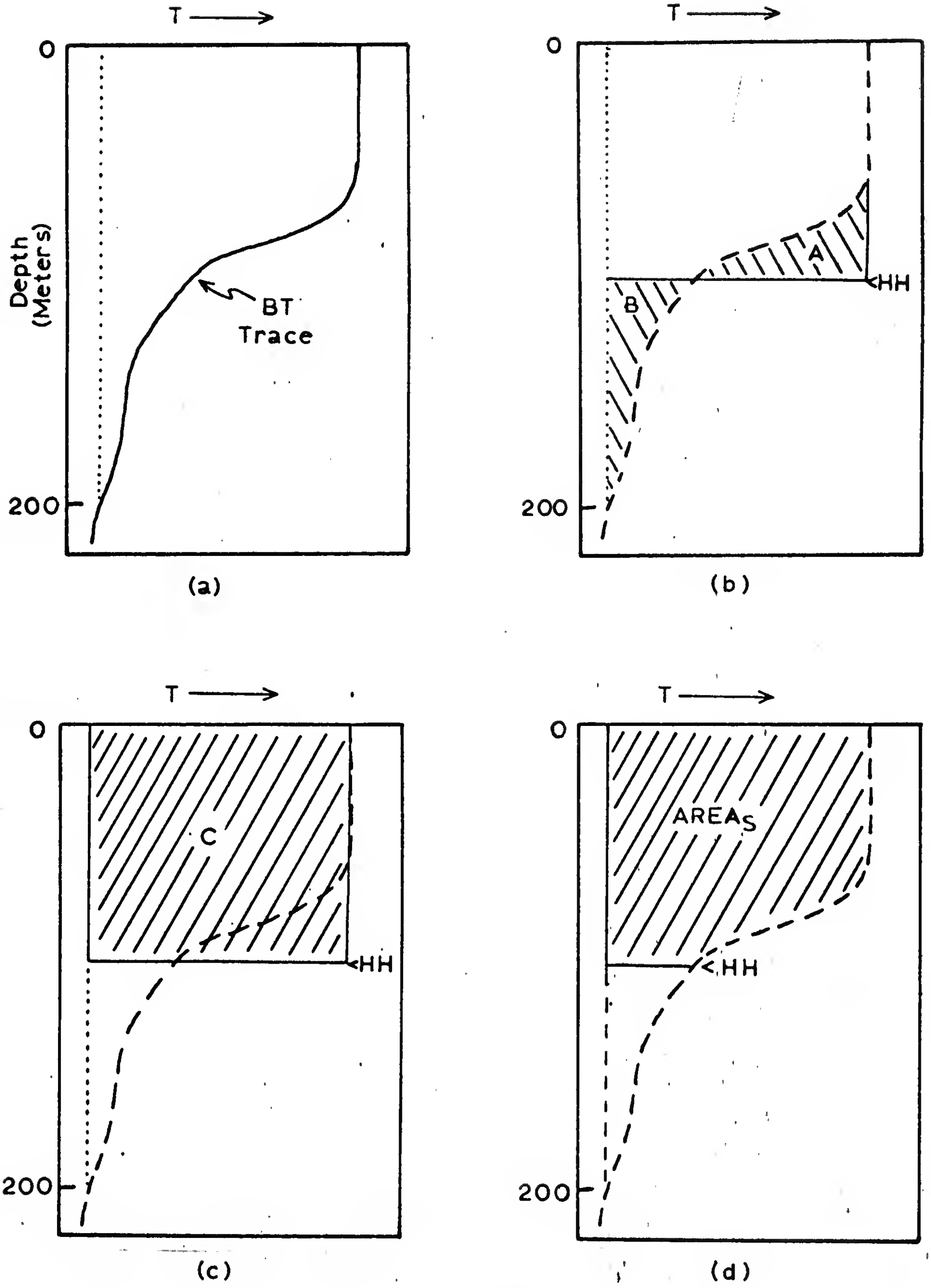


Figure 9

Representation of Area$_S$ Used in Determining $Q_S$

TABLE 6

VALUES FOR THE COEFFICIENT OF THERMAL EXPANSION BASED ON A SALINITY OF 32.5$^{o}/_{oo}$

| Temperature ($^{o}$C) | $\alpha \times 10^6$ (1/$^{o}$C) |
|---|---|
| $>$ 14.0 | 205 |
| 14.0 - 13.6 | 200 |
| 13.5 - 13.1 | 195 |
| 13.0 - 12.6 | 190 |
| 12.5 - 12.1 | 186 |
| 12.0 - 11.6 | 182 |
| 11.5 - 11.1 | 178 |
| 11.0 - 10.6 | 174 |
| 10.5 - 10.1 | 168 |
| 10.0 - 9.6 | 163 |
| 9.5 - 9.1 | 158 |
| 9.0 - 8.6 | 152 |
| 8.5 - 8.1 | 147 |
| 8.0 - 7.6 | 141 |
| 7.5 - 7.1 | 135.5 |
| 7.0 - 6.0 | 124.5 |
| $<$ 6.0 | 113 |

# APPENDIX IV

## FORECASTING METHOD PROPOSED BY KITAIGORODSKY

Kitaigorodsky [1], using similarity theory as described by Monin and Obukhov [14], developed an equation for forecasting the MLD. In developing the equation Kitaigorodsky assumed that thermal convection was negligible and that the vertical gradient of salinity was zero. This limits the use of the equation to the warm season when a stable layer exists. The equation does incorporate parameters that represent all the processes and variables that were considered in section 3 to cause or affect the MLD at OWS Papa.

Equation developed by Kitaigorodsky using similarity theory:

$$MLD = P \frac{T_a^{3/2}}{Q_F g \beta} \tag{8}$$

where: $T_a$ = a parameter proportional to the tangential stress of the wind on the sea surface

$Q_F$ = rate of heat flow from the atmosphere (LT/t);

$g$ = acceleration of gravity ($L/t^2$);

$\beta$ = coefficient of thermal expansion of sea water (1/T);

$P$ = a dimensionless coefficient that is a function of N; both N and P are determined empirically;

$N = \frac{T_a \Omega}{Q_F g \beta}$ (N is dimensionless);

$\Omega$ = coriolis parameter (1/t).

The value of P in equation (8) is given by a plot of 14 corresponding values of P and N, obtained by Kitaigorodsky from data from the

NORPAC ATLAS [15]. The large areal extent which the data represents makes the results widely applicable throughout the ocean during the warm season.

The present author, using data from OWS Papa [16] has determined 11 more pairs of values of P and N. Each of these 11 pairs represents from four to six BTs averaged together in an attempt to smooth out internal wave effects. The paired values of P and N given by both Kitaigorodsky and the present author are plotted on figure 10.

In calculating P and N, it appears on the basis of the reference [17] that Kitaigorodsky used an average (climatological) heat flow to find $Q_F$. The present author also used an average heat flow, the average heat flow since the time that the upper layer of the ocean was last isothermal. This average heat flow was arrived at in three steps.

1. Calculate the heat added in the surface layer since warming began. The method described for $Q_T$ in Appendix III was used for this calculation.
2. Determine the length of time elapsed since warming began.
3. Divide value determined in 1 above by the value determined in 2 above.

Despite the different locations of the NORPAC and OWS Papa data and despite the differences in $Q_F$, both sets of the points P versus N appear to fit the same curve (in figure 10). Still both methods for obtaining $Q_F$ seem weak. In section 4, $Q_F$ is replaced by another parameter to represent stability in order to eliminate that weakness.

Another problem in applying the method proposed by Kitaigorodsky lies in the steepness of the slope, $\frac{dP}{dN}$, for the small values of N.

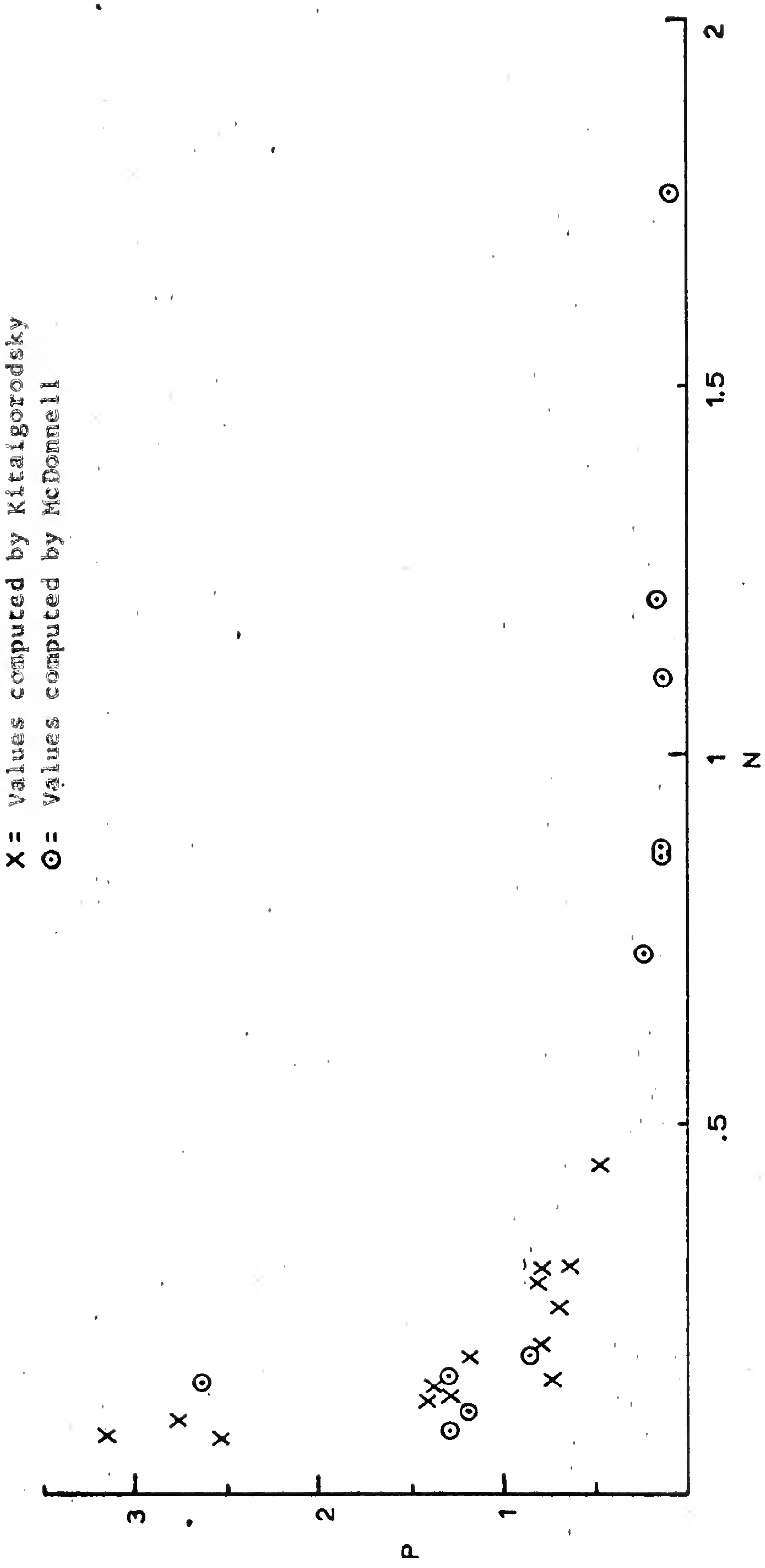


Figure 10
Values of P versus N Computed Using Equations Developed by Kitaigorodsky

To avoid these problems, the author derived by similarity methods an equation with more readily-available parameters and a more stable P(N).